Exercise 18

Prove the identity.

$$\frac{1+\tanh x}{1-\tanh x} = e^{2x}$$

Solution

Use the definitions listed on page 259.

$$\begin{aligned}
\frac{1+\tanh x}{1-\tanh x} &= \frac{1+\left(\frac{\sinh x}{\cosh x}\right)}{1-\left(\frac{\sinh x}{\cosh x}\right)} \\
&= \frac{1+\frac{\sinh x}{\cosh x}}{1-\frac{\sinh x}{\cosh x}} \times \frac{\cosh x}{\cosh x} \\
&= \frac{\cosh x + \sinh x}{\cosh x - \sinh x} \\
&= \frac{\left(\frac{e^x+e^{-x}}{2}\right)+\left(\frac{e^x-e^{-x}}{2}\right)}{\left(\frac{e^x+e^{-x}}{2}\right)-\left(\frac{e^x-e^{-x}}{2}\right)} \\
&= \frac{\frac{1}{2}e^x+\frac{1}{2}e^{-x}+\frac{1}{2}e^x-\frac{1}{2}e^{-x}}{\frac{1}{2}e^x+\frac{1}{2}e^{-x}-\frac{1}{2}e^x+\frac{1}{2}e^{-x}} \\
&= \frac{e^x}{e^{-x}} \\
&= \frac{e^x}{\frac{1}{e^x}} \times \frac{e^x}{e^x} \\
&= \frac{e^{2x}}{1} \\
&= e^{2x}
\end{aligned}$$